Average detour *D*-eccentricities of graphs

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Abstract

The average distance in a graph is one of the important index which can be used in many applications. Eccentricity is the maximum distance from a vertex to any other vertex of the graph. The average eccentricity of a graph is applicable in network theory etc. In this article, we study average eccentricity of graph using Detour D-distance. We derive some properties and compute it for some classes of graphs.

Key words: Detour *D*-distance, detour *D*-eccentricity, average detour *D*-eccentricity.

1. Introduction

In this paper we discuss a topological index, namely, average eccentricity of vertices w.r.t. detour *D*-distance. The concept of *D*-distance in graphs was introduced by Reddy Babu and Varma in [4]. The concept of detour *D*-distance in graphs was introduced by V. Venkateswara Rao and Varma in [5]. Gupta. S, et. al. [1] and Ghorbanifar M, [2] explained connective eccentric index. The authors introduced *D*-eccentric and detour *D*-eccentric connectivity index (see [6, 7]).

In this article, we calculate the average eccentricities of vertices using detour *D*-distance in some families of the graphs.

Throughout this paper, all the graphs we consider are assumed to be finite, simple and connected. For any unexplained notation, see [3]. We recall some definitions based on detour *D*-distance, see [5].

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The *detour D-distance*, $D^D(u,v)$, between two vertices u,v of a connected graph G is defined as $D^D(u,v) = \max\{l^D(u,v)\}$ if u and v are distinct and $D^D(u,v) = 0$ if $u = v$, where the maximum is taken over all $u - v$ paths s in G .

In a natural way, the *detour D-eccentricity* $e_D^D(v)$ of v is the detour D-distance to a farthest vertex from v. The *detour D-radius* , $r_D^D(G)$ and *detour D-diameter*, $daim_D^D(G)$, are defined as the minimum and maximum eccentricities, respectively. The *detour D-center*, $C_D^D(G)$, and *detour D-periphery*, $P_D^D(G)$, of graph G consists of the set of vertices of minimum and maximum eccentricity, respectively. A graph G is *detour D-self centered* if $V(G) = C_D^D(G)$.

The *average detour D-distance* between vertices is given by $\mu_D^D(G) = \frac{1}{n(n-1)} \sum_{\{u,v\}} D^D(u,v)$. The *detour D-distance matrix* of G, denoted by $M_D^D(G)$, is defined as $M_D^D(G) = [a_{i,j}]_{n\times n}$ where $a_{i,j} = D^D(u_i,u_j)$ is the detour D-distance between the vertices u_i and u_j. For any subset S of $V(G)$, we define detour D-distance between a vertex u and S as $D^D(u,S) = \max\{D^D(u,v) / v \in S\}$. Further, we have $\sigma_D^D(S) = \sum_{v\in V} D^D(v,S)$.

2. Results on average detour D-eccentricity

In a graph G, the *average detour D-eccentricity* of G is defined as $avec_D^D(G) = \frac{1}{|G|} \sum_{v_i \in V} e_D^D(v_i)$.

We begin with a result on detour D-self-centered graphs which is obvious.

Theorem 2.1 *For any graph, G,* $D^D - radius \le avec_D^D(G) \le D^D - diameter$.

Proof**:** From the definition, minimum eccentricity is detour D-radius and maximum eccentricity is detour D-diameter. Hence $D^D - radius \le avec_D^D(G) \le D^D - diameter$.

Theorem 2.2 *For detour D-self centered graph, G,* $D^D - radius = avec_D^D(G) = D^D - diameter$.

Proof**:** From the definition of detour D-self centered graph, detour D-radius and detour D-diameter are same. Hence $D^D - radius = avec_D^D(G) = D^D - diameter$.

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Theorem 2.3 *For any graph,* G*,* $avec_D^D(G) \geq \mu_D^D(G)$.

Proof**:** Let G be a graph with n vertices. Then the detour D-distance matrix will have n rows. The average detour D-distance of each row is less than or equal to detour D-eccentricity of the row, i.e., $\frac{1}{n-1}\sum_{j=1}^{n} a_{i,j} \leq e^D(v_i)$ for each i. Here, $a_{i,j}$ stands for the detour D-distance between the vertices v_i and v_j. Then taking the sum over all i, we get $\frac{1}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} \leq \sum_{i=1}^{n} e_D^D(v_i)$ .Then $\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} \leq \frac{1}{n}\sum_{i=1}^{n} e_D^D(v_i)$. Hence $\mu_D^D(G) \leq avec_D^D(G)$.

Theorem 2.4 *Let* G *be a graph with* n *vertices. Then* $avec_D^D(G) \leq r_D^D(G) + \frac{1}{n}\left[\sigma_D^D\left(C_D^D(G)\right)\right]$.

Proof**:** Let $u_1, u_2, \cdots, u_n$ be the vertices of G. Some of the $u_i - u_j$ detour paths will pass through the detour D-center and some may not.

Let P be a $u_i - u_j$ path in G passing through a detour D-central vertex u_k such that $D^D(u_i, u_k)$ is equal to detour D-radius, i.e., $D^D(u_i, u_k) = r_D^D(G)$. Also $C_D^D(G)$ is the set which contains all the central vertices. Clearly from the definition, detour D-distance from a vertex to $C_D^D(G)$ is always greater than detour D-distance from that vertex to the vertex u_j, i.e., $D^D\left(C_D^D(G), u_j\right) \geq D^D(u_i, u_j)$. Using triangle inequality we have $D^D(u_i, u_j) \leq D^D(u_i, u_k) + D^D(u_k, u_j) \leq r_D^D(G) + D^D\left(C_D^D(G), u_j\right)$.

Suppose $u_1^*, u_2^*, \cdots, u_n^*$ denote the detour D-eccentricity vertices of $u_1, u_2, \cdots, u_n$ resply. Then $e_D^D(u_i) \leq r_D^D(G) + D^D\left(C_D^D(G), u_i^*\right)$. Further, $avec_D^D(G) = \frac{1}{n}\left[e_D^D(u_1) + e_D^D(u_2) + \cdots + e_D^D(u_n)\right]$

$$\leq \frac{1}{n}\left[r_D^D(G) + D^D\left(C_D^D(G), u_1^*\right) + r_D^D(G) + D^D\left(C_D^D(G), u_2^*\right) + \cdots + r_D^D(G) + D^D\left(C_D^D(G), u_n^*\right)\right]$$

$$\leq \frac{1}{n}\left[n\, r_D^D(G) + \sum_{i=1}^{n} D^D\left(C_D^D(G), u_i^*\right)\right] \leq r_D^D(G) + \frac{1}{n}\sum_{i=1}^{n} D^D\left(C_D^D(G), u_i^*\right).$$

Hence $avec_D^D(G) \leq r_D^D(G) + \frac{1}{n}\left[\sigma_D^D\left(C_D^D(G)\right)\right]$.

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Theorem 2.5 *Let H be a spanning subgraph of a graph G* we have $avec_D^D(G) \geq avec_D^D(H)$.

Proof**:** If H be the spanning subgraph of the graph G then $H \subseteq G$ with same number of vertices and $E(H) \subseteq E(G)$. Clearly, as the number of edges may reduce in *H*, we have $\deg_G(v) \geq \deg_H(v)$. Hence $avec_D^D(G) \geq avec_D^D(H)$.

3. Average detour *D*-eccentricity of some families of graphs

Next, in this section, we calculate the average detour *D*-eccentricity of some classes of graphs. For these computations, we use the detour *D*-eccentricities of vertices of some graphs, which can be found in [5].

Theorem 3.1 *The average detour D-eccentricity of the complete graph, K_n, is $n^2 - 1$.*

Proof**:** In a complete graph K_n, each vertex is of degree $n-1$. Thus eccentricity of each vertex is $n^2 - 1$. Hence the average detour D-eccentricity $avec_D^D(K_n) = \frac{1}{|K_n|}\sum_{v_i \in V} e_D^D(v_i) = \frac{1}{n}\sum_{i=1}^{n} e_D^D(v_i)$

$= \frac{1}{n}\left(n\, e_D^D(v_i)\right) = e_D^D(v_i) = n^2 - 1.$

Theorem 3.2 *For the cycle graph C_n, the average detour D-eccentricity is $3n-1$.*

Proof: In the cycle graph C_n with n vertices, we have detour D-eccentricity of is $3n-1$. Then the average detour D-eccentricity is $avec_D^D(C_n) = \frac{1}{|C_n|}\sum_{v_i \in V} e_D^D(v_i) = \frac{1}{n}\sum_{i=1}^{n} e_D^D(v_i) = \frac{1}{n}\left(n\, e^D(v_i)\right) =$ $e_D^D(v_i)$. Thus $avec^D(C_n) = 3n-1$.

Theorem 3.3 *The average detour D-eccentricity of the wheel graph $W_{1,n}$ with $n+1$ vertices is $5n$.*

Proof: In the wheel graph $W_{1,n}$ with $n+1$ vertices let v_0 be the vertex which is adjacent to all other vertices. Then $\deg(v_0) = n$ and detour D-eccentricity is $5n$. All the remaining n vertices have degree 3 and D-eccentricity is $5n$. Thus the average detour D-eccentricity of the wheel graph $W_{1,n}$ is $avec_D^D(W_{1,n}) = \frac{1}{|W_{1,n}|}\sum_{v_i \in V} e_D^D(v_i) = \frac{1}{n+1}\left[\sum_{i=1}^{n} e_D^D(v_i) + 5n\right] = \frac{1}{n+1}\left[\sum_{i=1}^{n} 5n + 5n\right] = 5n.$

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Theorem 3.4 *Let* $K_{m,n}\left(m<n\right)$ *be the complete bipartite graph with* $m+n$ *vertices. Then the average detour D-eccentricity is* $\dfrac{m\left(m+n\right)^2+m^2\left(n+3\right)+n\left(2m-1\right)}{m+n}$.

Proof: In the complete bipartite graph $K_{m,n}\left(m<n\right)$ with $m+n$ vertices, the vertex set V can be partitioned as $V_1\cup V_2$ with V_1 contains m vertices and V_2 contains n vertices. The detour D-eccentricity of all vertices in V_1 is $m^2+mn+3m$ and detour D-eccentricity of all vertices in V_2 is $m^2+mn+2m-1$. Then the average detour D-eccentricity is $avec^D\left(K_{m,n}\right)=\dfrac{1}{\left|K_{m,n}\right|}\sum_{v_i\in V}e^D\left(v_i\right)$

$$=\frac{1}{m+n}\sum_{i=1}^{m+n}e^D\left(v_i\right)=\frac{1}{m+n}\left[\left(m\left(m^2+mn+3m\right)+n\left(m^2+mn+2m-1\right)\right)\right]$$

$$=\frac{m\left(m+n\right)^2+m^2\left(n+3\right)+n\left(2m-1\right)}{m+n}.$$

Theorem 3.5 *For the complete bipartite graph,* $K_{m,m}$*, with* $2m$ *vertices, the average detour D-eccentricity is* $2m^2+2m-1$.

Proof: In the complete bipartite graph $K_{m,m}$ with $2m$ vertices, the vertex set V can be partitioned as $V_1\cup V_2$ with V_1 contains m vertices and V_2 contains m vertices. Then in $K_{m,m}$ degree of each vertex is m and D-eccentricity of each vertex $2m^2+2m-1$. Thus the average detour D-eccentricity is

$$avec_D^D\left(K_{m,m}\right)=\frac{1}{\left|K_{m,m}\right|}\sum_{v_i\in V}e_D^D\left(v_i\right)=\frac{1}{2m}\left(\sum_{i=1}^{2m}(2m^2+2m-1)\right)=2m^2+2m-1.$$

Theorem 3.6 *For the Path graph,* P_n*, the average detour D-eccentricity of is* $\dfrac{1}{2n}\left(3n^2+8n-16\right)$ *if* n *is even and* $\dfrac{1}{2n}\left(3n^2+5n-10\right)$ *if* n *is odd.*

Proof: In the path graph P_n with n vertices, the two end vertices have degree 1 and detour D-eccentricity $3\left(n-1\right)$. The remaining $\left(n-2\right)$ vertices have degree 2 and detour D- eccentricity

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$\frac{3n-1}{2}$ if n is odd and $\frac{3n+2}{2}$ if n is even. We calculate average detour D-eccentricity by consider the even and odd cases separately.

Case (i) n is odd

The average detour D-eccentricity is

$$avec_D^D(P_n) = \frac{1}{|P_n|}\sum_{v_i \in V} e_D^D(v_i) = \frac{1}{n}\left[3(n-1) + \sum_{i=2}^{n}\frac{3n-1}{2} + 3(n-1)\right] = \frac{1}{2n}\left(3n^2 + 5n - 10\right).$$

Case (ii) n is even

The average detour D-eccentricity is

$$avec_D^D(P_n) = \frac{1}{|P_n|}\sum_{v_i \in V} e_D^D(v_i) = \frac{1}{n}\left[3(n-1) + \sum_{i=2}^{n}\frac{3n+2}{2} + 3(n-1)\right] = \frac{1}{2n}\left(3n^2 + 8n - 16\right).$$

Theorem 3.7 *For the Star graph,* $St_{1,n}$, *with* $n+1$ *vertices the average detour D-eccentricity is* $\frac{n^2+5n+2}{n+1}$.

Proof: In the star graph $St_{1,n}$, the degree of central vertex is n and detour D-eccentricity is $n+2$. All the remaining n vertices have degree 1 and detour D-eccentricity $n+4$. Thus the average detour D-eccentricity is

$$avec_D^D(St_{1,n}) = \frac{1}{|St_{1,n}|}\sum_{v_i \in V} e_D^D(v_i) = \frac{1}{n+1}\sum_{i=1}^{n+1} e_D^D(v_i) = \frac{1}{n+1}\left((n+2) + n(n+4)\right) = \frac{n^2+5n+2}{n+1}.$$

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