Complex dynamics and its stabilization in a predator prey model with the effect of refugia

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Abstract

The analysis of predator-prey relationship has become the focus of challenging scientific research in the field of mathematical and ecological studies. Our present work concerns food chain model with refugia in prey and intermediate predator species. The impacts of refuges, affecting population dynamics, are quite complicated in nature. Refugia influence relationship between predator and prey through interactions of refuse and non-refuse areas. We investigate local and global stability analysis of our system around the different equilibrium points under some conditions. In order to analyze the global nature of the model we perform detailed numerical experiments. According to our observation, the system exhibits different dynamical phenomena like stable focus, limit cycle oscillation, period-2 oscillation and chaos. For studying the effect of refugia with regard to chaotic dynamics we vary the parameters m_1 and m_2. We found that when the value of refugia parameter increases, our system goes down into stable focus through chaotic motion, period-2 and limit cycle oscillation. So, we may conclude that the refugia parameter has a stabilizing effect and we can use this as a chaotic nature controlling parameter in a dynamical system.

Keywords: Refugia in intermediate predator, chaotic motion, stable focus, half saturation constant and global stability.

1 Introduction

Predator-prey models with disease are a major concern and are now becoming an attractive field as eco-epidemiology. The study of interactions between predator and prey has been much progress and many mathematical and ecological problems remain challenging to us. The works of Lotka[14] and Volterra[20] are the first pioneering work about predator-prey interaction dynamics. Also, the remarkable work of Kermack-Mckendric[12] about epidemiology acquired many attractions among applied mathematicians, scientists and ecologists. Many researchers[1, 2, 3, 4, 17, 19, 22] have studied eco-epidemiological models introducing various factors, like harvesting, allee effects, refugia etc and developed the study of predator-prey relationship.

In nature, all living species want to live with own conditions. Every species likes a suitable safe environment, where it can live freely and reproduce. For this reason, prey populations usually move to areas where they are not threatened by their predators. Such refugia are engaged in two important roles - (a) serving to minimize the chance of extinction (due to predation), and (b) to damp prey-predator oscillations. Now, the way the prey refuges affect the population dynamics, is quite complex in nature. The relationship between predator and prey can be affected by refugia through movements between refuse and non-refuse areas. The mathematical models and a number of experiments indicate that refugia have stabilizing effects[6, 7, 9, 10] on predator-prey relations. Therefore, prey refuge becomes one of the key factor in predator-prey dynamics and many researchers[13, 15, 16, 18] have studied it. Recently, a diffusive predator-prey system is studied by Yang et al[23] and the effects of prey refuge and gestation delay are discussed. They have also investigated the nature of stability of the fixed points and bifurcation analysis of the model. Wang et al[21] showed that the prey species are able to refuge their predators using some techniques. Kar[11] analyzed the activities of harvesting and prey refuge by studing a prey-predator model. Chen et al[5] proved that the densities of prey and predator populations can be affected by introducing prey refuge in a system. To the best of our knowledge

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none of the studies was done about the joint effects of refugia on prey species and intermediate predator species in food chain model. But both the refugia factors have remarkable effects on the dynamics of food chain system. Here, we examine the effects of refugia on prey species and intermediate predator species in a prey-predator model. We have discussed the different aspects of our model and showed that the refugia in both the species are able to control chaotic dynamics of the system.

The paper is structured in the following systematic way. Firstly, we formed our mathematical model with some basic ideas in section(2). We have done stability analysis of fixed points or equilibrium points in section(3). We perform numerical analysis of our model through section (4). At the end, a conclusion of our study is given.

2 Formulation of the Model

Here we discuss the impact of refugia on food chain system pursuing Hastings and Powell [8] model(HP model) with type-II functional response(Holling) introducing prey refugia and intermediate predator refugia. With above assumptions our proposed model takes the following form:

$$
\begin{aligned}
\frac{dU}{dT} &= R_0U\left(1-\frac{U}{K_0}\right)-\frac{C_1A_1(1-m_1)VU}{B_1+(1-m_1)U}\\
\frac{dV}{dT} &= \frac{A_1(1-m_1)VU}{B_1+(1-m_1)U}-\frac{A_2(1-m_2)VW}{B_2+(1-m_2)V}-D_1V\\
\frac{dW}{dT} &= \frac{C_2A_2(1-m_2)VW}{B_2+(1-m_2)V}-D_2W.
\end{aligned}
\tag{1}
$$

Here U denotes prey species at lower level of food chain, V denotes intermediate predator species which preys U, and W refers top-predator species that preys V. Here, time is represented by T. The parameters R_0 and K_0 respectively refer intrinsic growth rate and carrying capacity of U species . The parameters C_1^{-1} and C_2 represent conversion rates of prey to predator for species V and W respectively; D_1 and D_2 indicates respectively constant death rates for species V and W. Maximal predation rates and half saturation constants for V and W species respectively denoted by constants A_i and $B_i (i = 1, 2)$. Here m_1 and m_2 are the refugia parameters for prey and intermediate predator populations.

Hastings and Powell [8] showed that the dynamic relation between prey and predator for a simple three-species food chain model is chaotic in a certain region of parametric space. The initial conditions for the system (1) are as follows:

$$U(0) > 0,\ V(0) > 0,\ W(0) > 0.$$

Now, we combine the parameters of the system (1) that control system behavior and we dimensionalize the system as follows:

$$u = \frac{U}{K_0}, \quad v = \frac{C_1 V}{K_0}, \quad w = \frac{C_1 W}{C_2 K_0} \ \text{ and } \ t = R_0 T$$

After some simplification we can write system (1) in the following form:

$$\begin{aligned}
\frac{du}{dt} &= u(1-u) - \frac{r_1(1-m_1)uv}{1+(1-m_1)k_1 u} \\
\frac{dv}{dt} &= \frac{r_1(1-m_1)uv}{1+(1-m_1)k_1 u} - \frac{r_2(1-m_2)vw}{1+(1-m_2)k_2 v} - n_1 v \\
\frac{dw}{dt} &= \frac{r_2(1-m_2)vw}{1+(1-m_2)k_2 v} - n_2 w
\end{aligned} \tag{2}$$

where

$$r_1 = \frac{A_1 K_0}{R_0 B_1}, \quad k_1 = \frac{K_0}{B_1}, r_2 = \frac{C_2 A_2 K_0}{C_1 R_0 B_2}, \quad k_2 = \frac{K_0}{C_1 B_2}, n_1 = \frac{D_1}{R_0}, \quad n_2 = \frac{D_2}{R_0}.$$

3 Stability analysis of the model

3.1 Analysis of local stability of the fixed points

Now, the system(2) has four fixed points or equilibrium points. Here, $F_0(0,\ 0,\ 0)$ and $F_1(1,\ 0,\ 0)$ denotes trivial equilibrium point and predator free equilibrium point respectively. This two fixed points exist for all values of the parameters. Again, $F_2(\bar{u},\ \bar{v},\ 0)$ denotes top predator free equilibrium point with

$$\bar{u} = \frac{n_1}{(r_1 - k_1 n_1)(1 - m_1)} \text{ and } \bar{v} = \frac{(1-\bar{u})[1 + (1-m_1)k_1\bar{u}]}{r_1(1-m_1)}.$$

The existence conditions of $F_2(\bar{u},\ \bar{v},\ 0)$ are $r_1 - k_1 n_1 > 0$ and $1 - m_1 > 0$.

Here, $F^*(u^*,\ v^*,\ w^*)$ refers the interior equilibrium point with, $v^* = \frac{n_2}{(r_2 - k_2 n_2)(1-m_2)}$, $w^* = \frac{[1+(1-m_2)k_2 v^*][(1-u^*)u^* - n_1 v^*]}{(1-m_2)r_2 v^*}$ and u^* is given by the positive root of the following equation $k_1(1-m_1)u^{*2} + [1 - k_1(1-m_1)]u^* + A = 0$ with $A = \frac{n_2 r_1 (1-m_1)}{(r_2 - k_2 n_2)(1-m_2)} - 1$. The existence conditions of $F^*(u^*,\ v^*,\ w^*)$ are $r_2 - k_2 n_2 > 0$ and $1 - m_2 > 0$.

Now, we calculate the Jacobian matrix of our system (2) around an arbitrary point $(u,\ v,\ w)$. Let, this Jacobian is denoted by $J = (J_{ij})_{3\times 3}$ where,

$$J_{11} = 1 - 2u - \frac{r_1(1-m_1)v}{[1 + k_1(1-m_1)u]^2},\ J_{12} = -u,\ J_{13} = 0$$

$$J_{21} = \frac{r_1(1-m_1)v}{[1 + k_1(1-m_1)u]^2},\ J_{22} = 0,\ J_{23} = -\frac{r_2(1-m_2)v}{1 + k_2(1-m_2)v}$$

$$J_{31} = 0,\ J_{32} = \frac{r_2(1-m_2)w}{[1 + k_2(1-m_2)v]^2},\ J_{33} = 0$$

.

Theorem 1. The equilibrium point $F_0(0,\ 0,\ 0)$ is unstable for all values of the parameters. Predator free equilibrium point $F_1(1,\ 0,\ 0)$ will be locally stable if $N_{01} < 1$ where $N_{01} = \frac{r_1(1-m_1)}{n_1[1+k_1(1-m_1)]}$. The top predator free equilibrium point $F_2(\bar{u},\ \bar{v},\ 0)$ will be locally asymptotically stable if $M_{11} + M_{22} < 0$, $M_{11}M_{22} - M_{12}M_{21} > 0$ and $N_{20} < 1$ where $N_{02} = \frac{r_2(1-m_2)\bar{v}}{n_2[1+k_2(1-m_2)\bar{v}]}$.

Proof. Let the Jacobian matrix evaluated at F_0 is denoted by J_0. The roots of the characteristic equation of the Jacobian matrix J_0 are 1, $-n_1$and $-n_2$. Now, one eigenvalue of the matrix J_0 is $1 > 0$, so this equilibrium point F_0 is unstable for all values of the parameters.

Again, let J_1 refers the Jacobian matrix of our system computed around the predator free equilibrium point F_1. The roots of the characteristic equation of the Jacobian matrix J_1 are -1, $\frac{r_1(1-m_1)}{1+k_1(1-m_1)} - n_1$, and $-n_2$.

Hence, we can say that the fixed point F_1 will be stable if $\frac{r_1(1-m_1)}{1+k_1(1-m_1)} - n_1 < 0$, that is $N_{01} < 1$ where $N_{01} = \frac{r_1(1-m_1)}{n_1[1+k_1(1-m_1)]}$.
Now, let $J_2 = (M_{ij})_{3\times 3}$ denotes the Jacobian matrix of system(2) evaluated around top predator free equilibrium point F_2, where,
$M_{11} = 1 - 2\bar{u} - \frac{r_1(1-m_1)\bar{v}}{[1+k_1(1-m_1)\bar{u}]^2}$, $M_{12} = -\frac{r_1(1-m_1)\bar{u}}{1+k_1(1-m_1)\bar{u}}$, $M_{13} = 0$
$M_{21} = \frac{r_1(1-m_1)\bar{v}}{[1+k_1(1-m_1)\bar{u}]^2}$, $M_{22} = \frac{r_1(1-m_1)\bar{u}}{1+k_1(1-m_1)\bar{u}} - n_1$, $M_{23} = -\frac{r_2(1-m_2)\bar{v}}{1+k_2(1-m_2)\bar{v}}$
$M_{31} = 0$, $M_{32} = 0$, $M_{33} = \frac{r_2(1-m_2)\bar{v}}{1+k_2(1-m_2)\bar{v}} - n_2$

The roots of the characteristic equation of the Jacobian matrix J_2 are $\frac{r_2(1-m_2)\bar{v}}{1+k_2(1-m_2)\bar{v}} - n_2$ and the roots of the equation

$$\lambda^2 - (M_{11} + M_{22})\lambda + (M_{11}M_{22} - M_{12}M_{21}) = 0 \tag{3}$$

Now, the roots of the equation(3) will be negative if $M_{11} + M_{22} < 0$ and $M_{11}M_{22} - M_{12}M_{21}) > 0$. Hence, from above discussion it is clear that top predator free equilibrium point $F_2(\bar{u},\ \bar{v},\ 0)$ will be locally asymptotically stable if $M_{11} + M_{22} < 0$, $M_{11}M_{22} - M_{12}M_{21} > 0$ and $M_{33} < 0$, i.e. $\frac{r_2(1-m_2)\bar{v}}{1+k_2(1-m_2)\bar{v}} - n_2 < 0$ or $N_{20} < 1$ where $N_{02} = \frac{r_2(1-m_2)\bar{v}}{n_2[1+k_2(1-m_2)\bar{v}]}$.

3.2 Analysis of local stability of Interior equilibrium point

Theorem 2. The interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ will be asymptotically stable if $V_{11} < 0$.
Proof. Let, the Jacobian matrix evaluated around interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ is given by

$$J^* = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}$$

Where,
$V_{11} = 1 - 2u^* - \frac{r_1(1-m_1)v^*}{[1+k_1(1-m_1)u^*]^2}$, $V_{12} = -\frac{r_1(1-m_1)u^*}{1+k_1(1-m_1)u^*}$, $V_{13} = 0$
$V_{21} = \frac{r_1(1-m_1)v^*}{[1+k_1(1-m_1)u^*]^2}$, $V_{22} = 0$, $V_{23} = -\frac{r_2(1-m_2)v^*}{1+k_2(1-m_2)v^*}$

$V_{31} = 0$, $V_{32} = \frac{r_2(1-m_2)w^*}{[1+k_2(1-m_2)v^*]^2}$, $V_{33} = 0$

The characteristic equation with respect to Jacobian matrix J^* can be written as-
$\theta^3 + \gamma_1\theta^2 + \gamma_2\theta + \gamma_3 = 0$
where,

$$\gamma_1 = -V_{11}, \quad \gamma_2 = -(V_{12}V_{21} + V_{23}V_{32}), \quad \gamma_3 = V_{11}V_{23}V_{32}.$$

Hence, the interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ will be stable asymptotically, if γ_1, γ_2 and γ_3 satisfy the Routh-Hurwitz conditions $(i)\gamma_1, \gamma_3 > 0$ and $(ii)\gamma_1\gamma_2 - \gamma_3 > 0$.

Now, since $m_1 < 1$ and $m_2 < 1$ thus all the above conditions will be satisfied if $V_{11} < 0$. Therefore we can conclude that the interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ will be stable asymptotically if $V_{11} < 0$.

3.3 Analysis of global Stability around Interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$

Theorem 3: The interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ with respect to system(2) will be globally stable(asymptotically) if the following conditions hold
$(i)v^* < \frac{D}{r_1k_1(1-m_1)^2}$
$(ii)v^* < u^* + k_1u^*(1-m_1)$
$(iii)\frac{r_1u^*(1-m_1)}{D} + \frac{r_1k_1uu^*(1-m_1)^2}{D} < n_1 + \frac{r_2w^*(1-m_2)}{D_1}$
$(iv)w^* < v^* + k_2vv^*(1-m_2)$
$(v)r_2v(1-m_2) + r_2k_2vv^*(1-m_2)^2 < D_1n_2$
Proof: To study the global stability nature of the system(2) around the interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ we consider the following positive definite Lyapunov function
$L(u,v,w) = u + u^*log(\frac{u^*}{u}) + \frac{1}{2}(v-v^*)^2 + \frac{1}{2}(w-w^*)^2$.
Now, calculating time derivative of $L(u,v,w)$, we get
$\frac{dL}{dt} = (u-u^*)[1-u-\frac{r_1v(1-m_1)}{1+k_1(1-m_1)u}] + (v-v^*)[\frac{r_1uv(1-m_1)}{1+k_1(1-m_1)u} - \frac{r_2vw(1-m_2)}{1+k_2(1-m_2)v} - n_1v] +$
$(w-w^*)[\frac{r_2vw(1-m_2)}{1+k_2(1-m_2)v} - n_2w]$
$= (u-u^*)[u^* + \frac{r_1v^*(1-m_1)}{1+k_1(1-m_1)u^*} - u - \frac{r_1v(1-m_1)}{1+n_1(1-m_1)u}] + (v-v^*)[\frac{r_1uv(1-m_1)}{1+n_1(1-m_1)u} -$

$\frac{n_2vw(1-m_2)}{1+k_2(1-m_2)v} - n_1v - \frac{r_1u^*v^*(1-m_1)}{1+k_1(1-m_1)u^*} + \frac{r_2v^*w^*(1-m_2)}{1+k_2(1-m_2)v^*} + n_1v^*] + (w-w^*)[\frac{r_2vw(1-m_2)}{1+k_2(1-m_2)v} - n_2w - \frac{r_2v^*w^*(1-m_2)}{1+k_2(1-m_2)v^*} + n_2w^*]$.

After some algebraic calculations, we obtain
$\frac{dL}{dt} = -[1 - \frac{r_1k_1v^*(1-m_1)^2}{D}](u-u^*)^2 - [\frac{r_1k_1u^*(1-m_1)^2}{D} + \frac{r_1u^*(1-m_1)}{D} - \frac{r_1v^*(1-m_1)}{D}](u-u^*)(v-v^*) - [n_1 - \frac{r_1u^*(1-m_1)}{D} - \frac{r_1k_1uu^*(1-m_1)^2}{D} + \frac{r_1w(1-m_2)}{D_1}](v-v^*)^2 - [\frac{r_2k_2vv^*(1-m_2)^2}{D_1} + \frac{r_2v^*(1-m_2)}{D_1} - \frac{r_2w^*(1-m_2)}{D_1}](w-w^*)(v-v^*) - [n_2 - \frac{r_2v(1-m_2)}{D_1} - \frac{r_2k_2vv^*(1-m_2)^2}{D_1}](w-w^*)^2$, where $D = [1+k_1(1-m_1)u][1+k_1(1-m_1)u^*]$ and $D_1 = [1+k_2(1-m_2)v][1+k_2(1-m_2)v^*]$.

Hence, when the conditions(stated above) of the theorem hold, then the above expression will be negative definite and the function $L(u, v, w)$ will be a Lyapunov function around the interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ which is positive definite. Now, we can conclude that, the system(2) with respect to interior equilibrium point $F^*(u^*,\ v^*,\ w^*)$ is globally asymptotically stable.

4 Numerical analysis of the model

To study the global dynamical behaviour of our proposed model detailed numerical simulations have been done using MATLAB software. There are two important parameters, one is the refugia parameter for prey species(m_1) and the other is the refugia parameter for intermediate predator(m_2). So, we will observe the role of both the refugia parameters m_1 and m_2. Firstly we notice the chaotic nature of our proposed system(Figure-1) for the set of parameters values $r_1 = 5.0, r_2 = 0.1, k_1 = 3.0, k_2 = 2.0, n_1 = 0.4, n_2 = 0.01, m_2 = 0.0$ and $m_1 = 0.08$. Now we vary the value of m_1 to observe different dynamical behaviours. From Figure(2a) we have observed the chaotic tea-cup attractor for $m_1 = 0.08$. When the value of m_1 is increased from 0.08 to 0.2 we have found the period-2 oscillation. Now we again increase the value of m_1 from 0.2 to 0.8 we obtain the limit cycle oscillation(Figure-2c) and finally we observe the stable focus oscillation for $m_1 = 0.85$(Figure-2d). From Figure-2, we notice that the system reaches into a stable focus state through various non-linear dynamics like chaos, period-2 oscillation and limit cycle oscillation. Now we explain the biological reason for stabilization of chaotic dynamics for increasing the refugia parameter m_1 for prey species. If we increase the value of m_1 many prey species escape from predation zone of intermediate predator and intermediate predator decreases due to want of scarcity of prey species. This type of competition helps the chaotic system move to stable state.

To observe global dynamical changes for variation of refugia parameter(m_1) a bifurcation diagram(Figure-3) is drawn. We have observed from this diagram that the system arrives at stable state through various nonlinear phenomena like chaotic motion, period-2 and limit cycle oscillation. We have already clearly distinguished between chaos, period-2 oscillation and limit cycle through phase diagram(Figure-2). In the Figure-3 we have found the complete dynamical behaviour of the system viz chaos to period-2, period-2 to limit cycle and limit cycle to stable focus. We draw this diagram using maximum point and minimum point of time series solution of the system. Here upper curve(red colour) indicates the maximum points and lower curve(blue curve) indicates the minimum points.

Another important parameter is m_2 which is the refugia parameter for intermediate predator. Now, the impact of this parameter on the chaotic dynamics will be observed. From Figure-4, we have noticed that the system exhibits chaotic time series solution for $m_1 = 0.0$, $m_2 = 0.06$ and other parameters fixed as in Figure-1. This figure shows the biologically that when refugia behaviour in the intermediate predator is low the system preserve the chaotic dynamics. Now we will increase the value of m_2 and observe the changes of chaos. From Figure-5(a) it is noticed the chaos for $m_2 = 0.06$. When the value of m_2 is increased from 0.06 to 0.1, chaos becomes period-2 oscillations(Figure-5b). Now, we again increase the value of m_2 from 0.1 to 0.15, we notice that the system goes down into limit cycle oscillation(Figure-5c). Finally, we observe the stable focus for $m_2 = 0.25$(Figure-5d). From Figure-5, we observe that the system enters into stable focus oscillations through different non-linear phenomena such as chaos, period-2 oscillation, limit cycle for increasing the value of m_2.

Now, we draw a bifurcation diagram for m_2 to observe the actual dynamical behaviour of our system by changing the value of m_2. From Figure-6, we found that the system enters into stable focus state through different nonlinear behaviour like chaotic motion, period-2 and limit cycle. Hence, we can decide that the refugia parameter m_2 give a stable food chain system.

Now, we will observe the simultaneous effect of both refugia parameter m_1 and m_2 on the chaotic dynamics. From Figure-7, we noticed that the system shows periodic limit cycle oscillation for $m_1 = 0.03$ and $m_2 = 0.02$. It is obvious that when both prey and intermediate predator species use refugia behaviour chaos goes down into limit cycle oscillation. From Figure-8 it is observed that the system enter into stable state for different combined values of m_1 and m_2. From Figure-8(a) we found that the system shows chaotic tea-cup attractor for $m_1 = 0.02$ and $m_2 = 0.03$. From Figure-8(b) we notice that the system exhibits period-2 oscillation for $m_1 = 0.08$ and $m_2 = 0.09$. Figure-8(c) shows the periodic limit cycle oscillation of the system for $m_1 = 0.2$ and $m_2 = 0.16$. At last, the system reaches at stable state for $m_1 = 0.32$ and $m_2 = 0.25$(Figure-8(d)). Further, we observed the system dynamics around interior equilibrium point and the dynamics of other equilibrium points for variation of m_1 and m_2 simultaneously. From Figure-9(a) it is observed tea cup chaotic attractor and the motion of the trajectories around different equilibrium points for $m_1 = 0.03$ and $m_2 = 0.02$. We have also found the period-doubling trajectory around the interior equilibrium point and all other trajectories around the different boundary equilibrium points converges to period-2 oscillations for $m_1 = 0.08$ and $m_2 = 0.09$ (Figure-9(b)). Again, for increasing values of m_1 and m_2, we notice the limit cycle oscillation around interior equilibrium points and the trajectories around the boundary equilibrium points converges to limit cycle trajectory $m_1 = 0.2$ and $m_2 = 0.16$(Figure-9(c)). Finally, we observe the stable focus motion for $m_1 = 0.32$ and $m_2 = 0.25$(Figure-9(d)) and from this figure it is also observed that the trajectories around the boundary equilibrium points go to stable focus. So, it is observed that the simultaneous effect of m_1 and m_2 on the chaotic dynamics is very important for biological conservation. However these two parameters give a sustained stable system.

From overall numerical studies we first analyzed the role of m_1 on the chaotic dynamics and noticed that m_1 plays an significant role in controlling the chaotic dynamics through different non-linear behaviour such as chaos, period-2 and limit cycle. We also noticed that refugia parameter m_2 able to control the chaotic dynamics and give a sustained stable food chain ecosystem through chaos, period-2 oscillation and limit cycle. Next we have observed the simultaneous effect of m_1 and m_2 on the dynamical behaviour of our proposed system. We found that both m_1 and m_2 can stabilize the chaotic dynamics.

5 Conclusion

In this work we have discussed the relationship between predator and prey in a food chain model introducing refugia effects. The nature of stability of the model around different equilibrium points has been analyzed. From our study, it is evident that refugia technique can be very useful to preserve species. Refugia strategies also help the species survive from extinction. This can be an effective measure in scientific conservation of endangered species. It can also be applied to manage and restore the balance of chaotic food chain or biota in an ecosystem. We believe that our study may open a new area of research in future.

References

[1] Anderson, R.M., May, R.M. 1986. The invasion, persistence and spread of infectious diseases within animal and plant communities. Philos. Trans. R. Soc. Lond. B, 314, 533-570.

[2] Arino, O., Abdllaoui, A., Mikram, J., Chattopadhyay, J., Infection on prey population may act as a biological control in ratio-dependent predator-prey model. *Nonlinearity*, 17, 1101-1116, (2004).

[3] Chattopadhyay, J., Arino, O., A predator-prey model with disease in the prey. *Nonlinear Analysis*, 36, 747-766, (1999).

[4] Chattopadhyay, J. and Bairagi, N., Pelicans at risk in Salton Sea-an eco-epidemiological model. *Ecological Modelling*, 136, 103-112, (2001).

[5] Chen, F., Chen, L., Xie, X.: On a Leslie-Gower predator-prey model incorporating a prey refuge. Nonlinear Anal., Real World Appl. 10(5), 2905-2908 (2009)

[6] Collings, J. B., 1995. Bifurcation and stability analysis of temperature-dependent mite predatorprey interaction model incorporating a prey refuge. Bull. Math. Biol, Vol. 57, 63-76.

[7] Freedman, H. I., 1980. Deterministic mathematical method in population ecology. Marcel Debber, New York.

[8] Hastings, A. and Powell, T., Chaos in three-species food chain. *Ecology*,72(3), 896-903, (1991).

[9] Hochberg, M. E., Holt, R. D., 1995. Refuge evolution and the population dynamics of coupled of host-parasitoid asssociations. Evolutionary Ecology, Vol. 9, 633-661.

[10] Huang, Y., Chen, F., Zhong, L., 2006. Stability analysis of preypredator model with Holling type-III response function incorporating a prey refuge. Appl. Math. Comput, Vol. 182, 672-683.

[11] Kar, T. K.: Modelling and analysis of a harvested prey-predator system incorporating a prey refuge. Journal of Computational and Applied Mathematics, vol. 185, no. 1, pp. 19-33, 2006.

[12] Kermack, W.O., McKendrick, A.G., Contributions to the mathematical theory of epidemics, part-1. *Proc. R Soc. London Ser.*, A115, 700-721, (1927).

[13] Krivan, V., 1998. Effect of optimal antipredator behaviour of prey on predatorprey dynamics: the role of refuge. Theor. Popul. Biol, Vol. 53, 131-142.

[14] Lotka, A.J., " Elements of Physical Biology", Baltimore: *Williaams and Wilkins Co. Inc.*, (1924).

[15] McNair, J. N., 1986. The effect of refuge on pre-predator interactions: a reconsideration. Theor. Popul. Biol, Vol. 29, 38-63.

[16] Ruxton, G.D., 1995. Short term refuge use and stability of predatorprey model. Theor. Popul. Biol, Vol. 47, 1-17

[17] Schaffer, W.M. and Kot, M., Chaos in ecological systems the coals that Newcastle forgot. *Trends in Ecology and Evolution*, 1, 58-63, (1986).

[18] Sih, A., 1987. Prey refuge and predatorprey stability. Theor. Popul. Biol, Vol. 31, 1-12.

[19] Venturino, E., Epidemics in predator-prey model: disease in the predators. *IMA J. Math. Appl. Med. Biol.*, 19, 185-205, (2002).

[20] Volterra, V.: Variazioni e fluttauazionidel numero d'individui in specie animals conviventi, Mem. Acad. Lincei, 2, 31-33. (Translation in an appendix to Chapmain's Animal Ecology, New York, 1931), (1926).

[21] Wang, H., Morrison, W., Singh, A., Weiss, H.: Modeling inverted biomass pyramids and refuges in ecosystems. Ecological Modelling, vol. 220, no. 11, pp. 1376-1382, 2009.

[22] Xiao, Y and Chen, L., Modelling and analysis of apredator-prey model with disease in the prey. *Mathematical biosciences*, 171, 59-82, (2001).

[23] Yang, R., Ren, H., Cheng, X.: A diffusive predator-prey system with prey refuge and gestation delay. Advances in Difference Equations (2017) 2017:158 DOI 10.1186/s13662-017-1197-z.

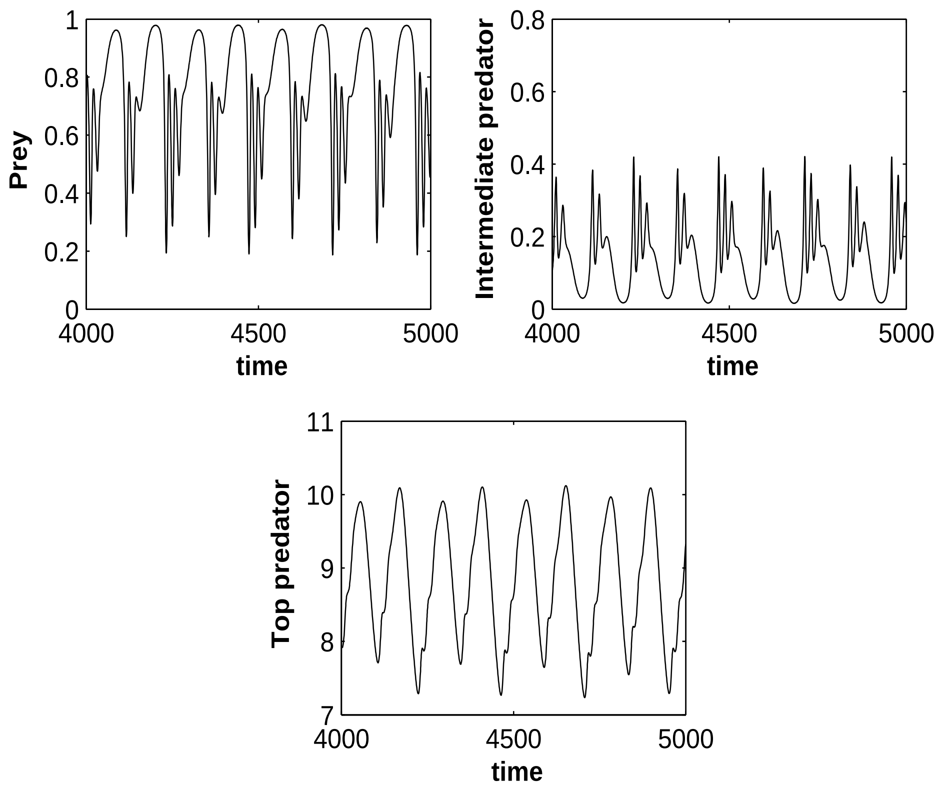

Figure 1: The system(2) shows chaotic solution for $r_1 = 5.0, r_2 = 0.1, k_1 = 3.0, k_2 = 2.0, n_1 = 0.4, n_2 = 0.01, m_2 = 0.0$ and $m_1 = 0.08$.

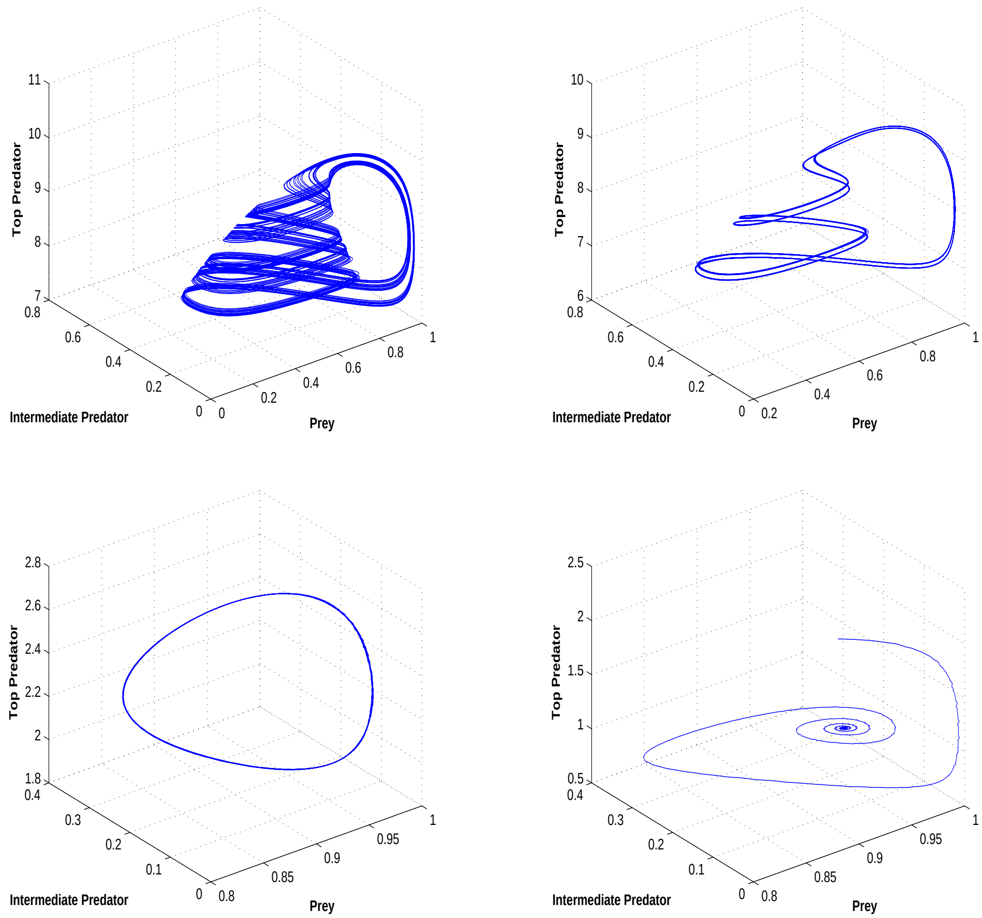

Figure 2: The figure (a) shows chaotic tea-cup for $m_1 = 0.08$, (b) exhibits the period-2 oscillation of the system for $m_1 = 0.2$, (c) shows the limit cycle oscillation for $m_1 = 0.8$. Finally, (d) describes the steady state stable distribution with $m_1 = 0.85$ and other parameter values given in Figure 1.

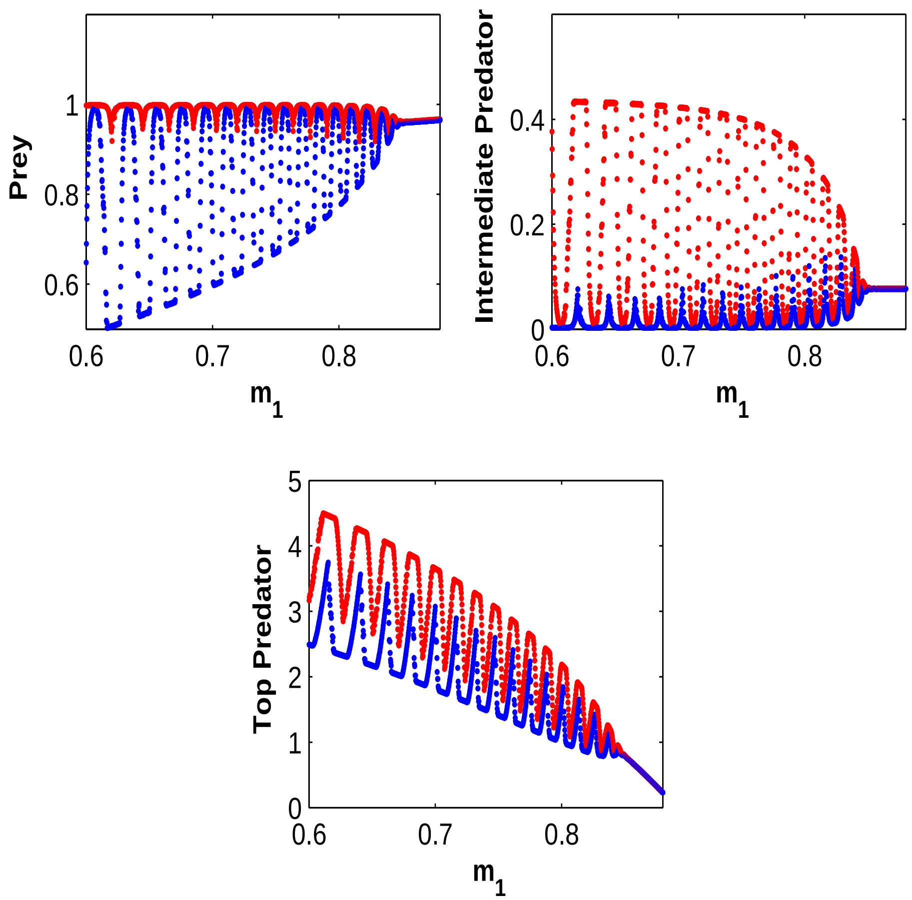

Figure 3: A bifurcation diagram is presented for $m_1 \in [0.6, 0.88]$ and other values of the parameters are given in Figure 1.

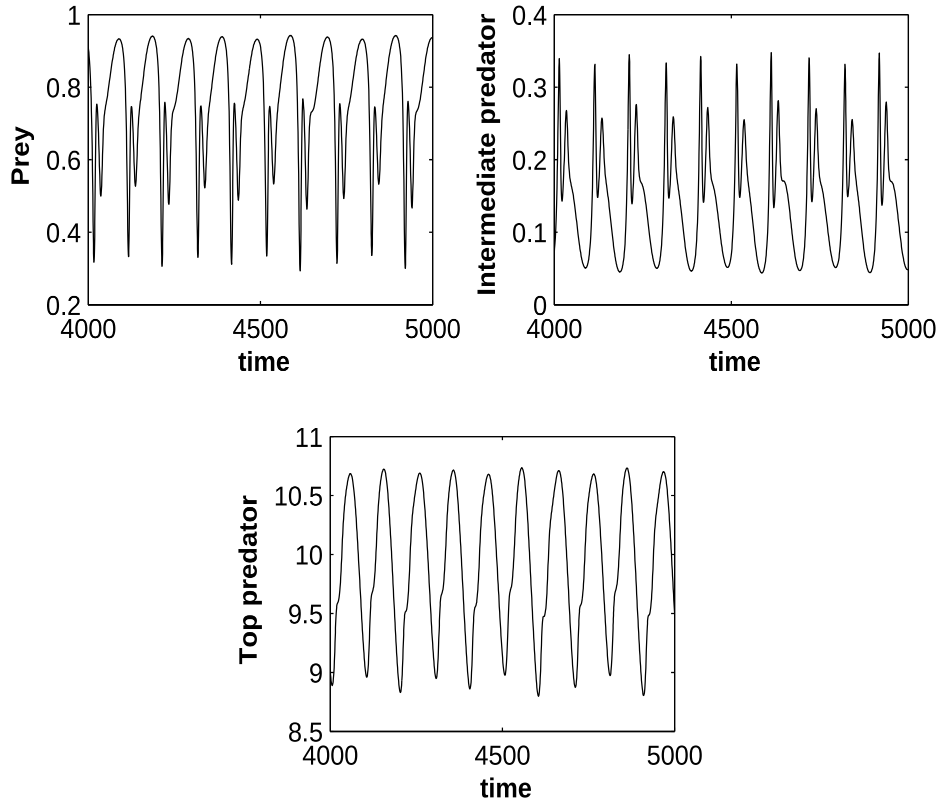

Figure 4: The Figure represents the chaotic solution of system (2) for $r_1 = 5.0, r_2 = 0.1, k_1 = 3.0, k_2 = 2.0, n_1 = 0.4, n_2 = 0.01, m_1 = 0.0$ and $m_2 = 0.06$.

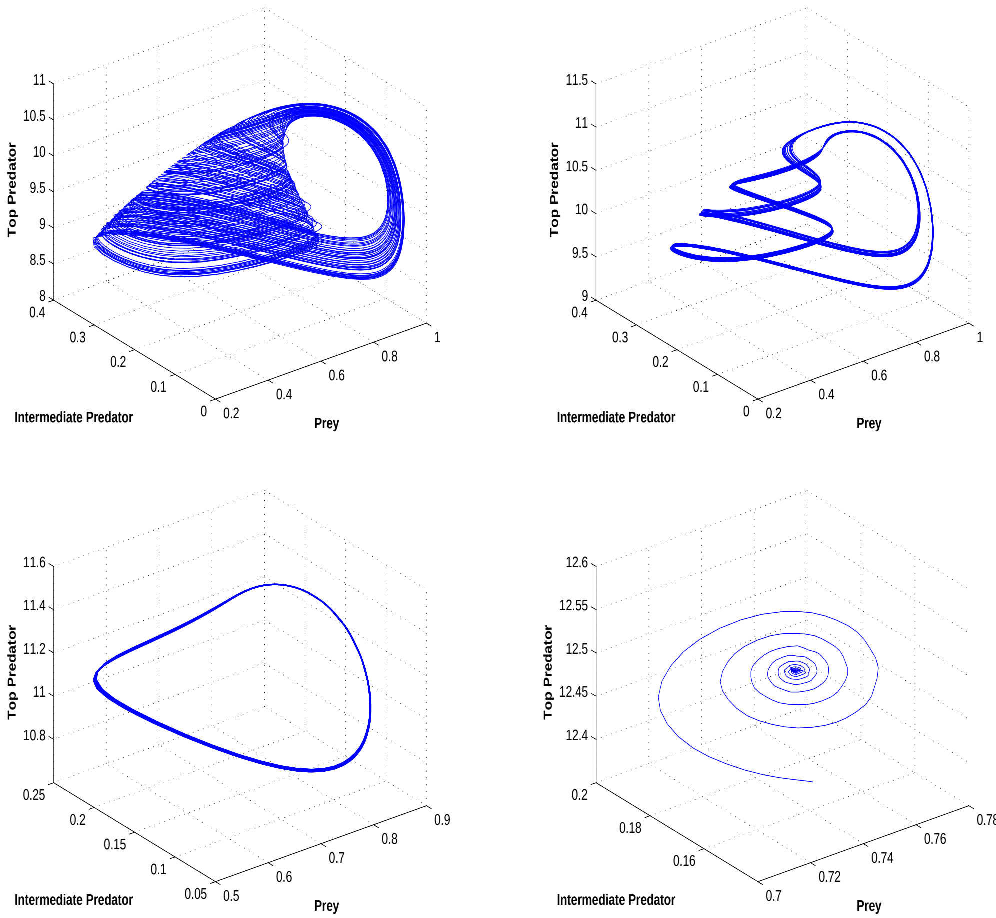

Figure 5: The Figure (a) shows chaotic tea-cup for $m_2 = 0.06$, (b) depicts the period-2 oscillation of the system for $m_2 = 0.1$, (c) shows the limit cycle oscillation for $m_2 = 0.15$. Finally, (d) describes the steady state stable distribution with $m_2 = 0.25$ and other parameter values given in Figure 1.

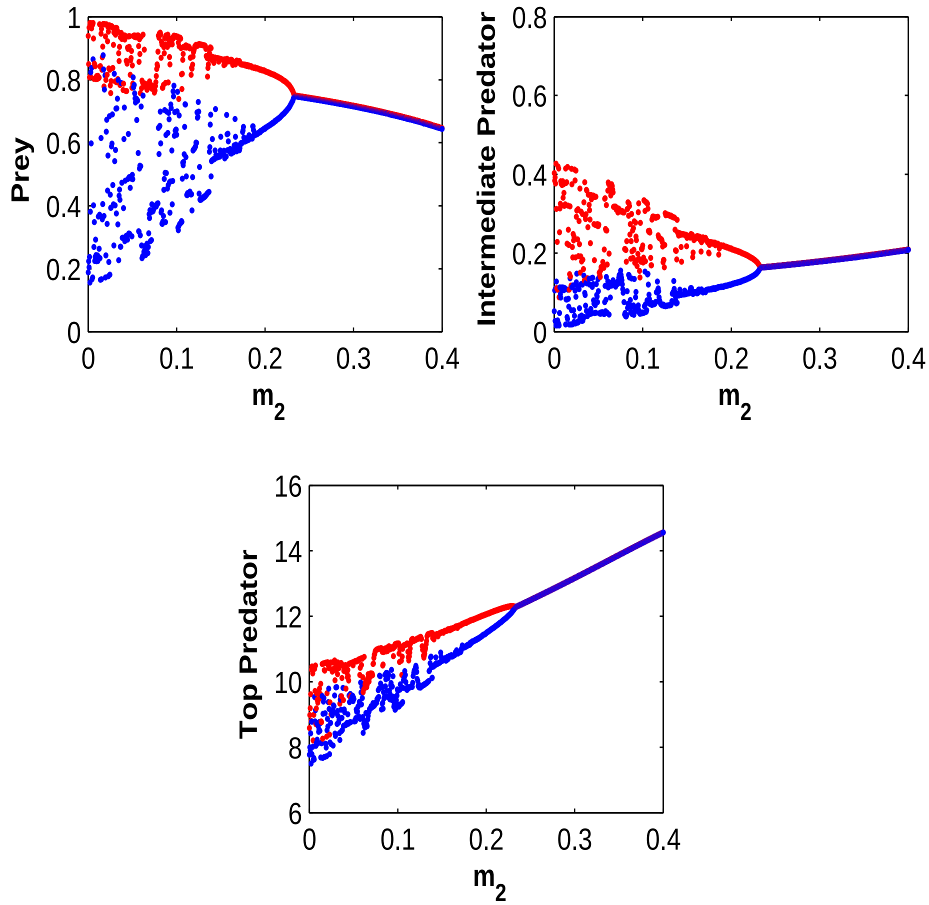

Figure 6: Figure shows bifurcation diagram for $m_2 \in [0.0, 0.4]$ and other values of the parameters are given in Figure 1.

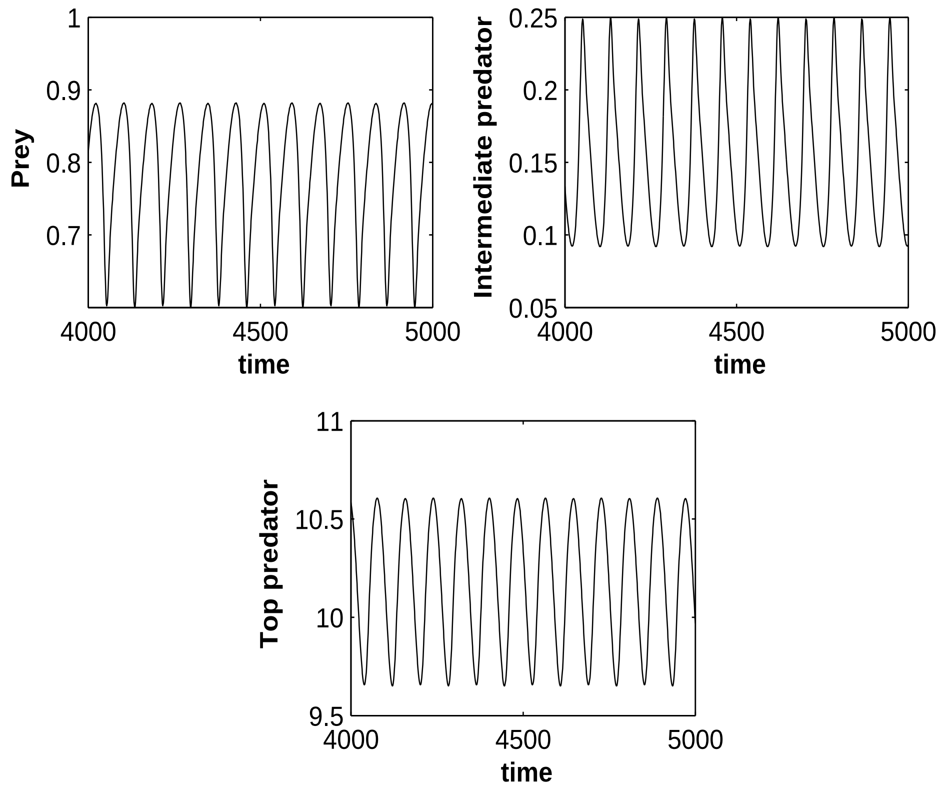

Figure 7: The Figure represents the chaotic solution of system (2) for $r_1 = 5.0, r_2 = 0.1, k_1 = 3.0, k_2 = 2.0, n_1 = 0.4, n_2 = 0.01, m_2 = 0.03$ and $m_1 = 0.02$.

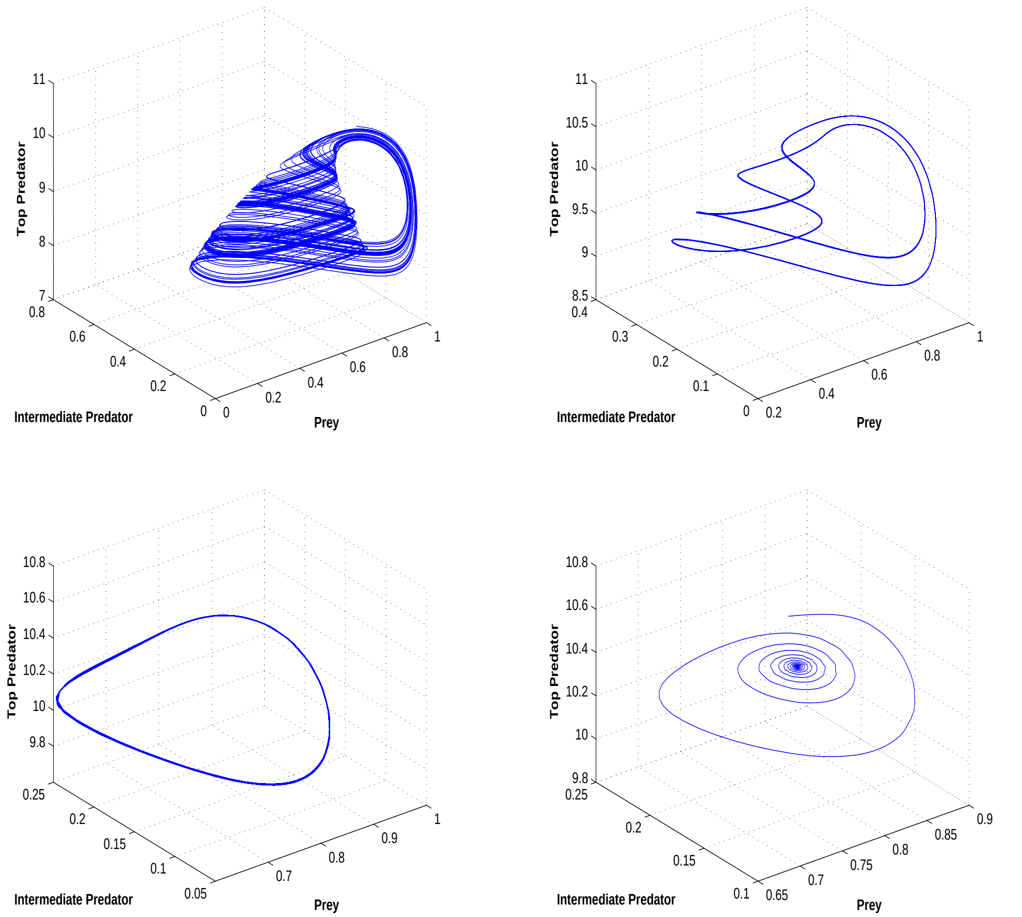

Figure 8: The Figure (a) shows chaotic tea-cup for $m_1 = 0.02$ and $m_2 = 0.03$, (b) shows the period-2 oscillation of the system for $m_1 = 0.08$ and $m_2 = 0.09$, (c) shows the limit cycle oscillation for $m_1 = 0.2$ and $m_2 = 0.16$. Finally, (d) shows steady stable state distribution with $m_1 = 0.32$ and $m_2 = 0.25$ and other values of the parameters are given in Figure 1.

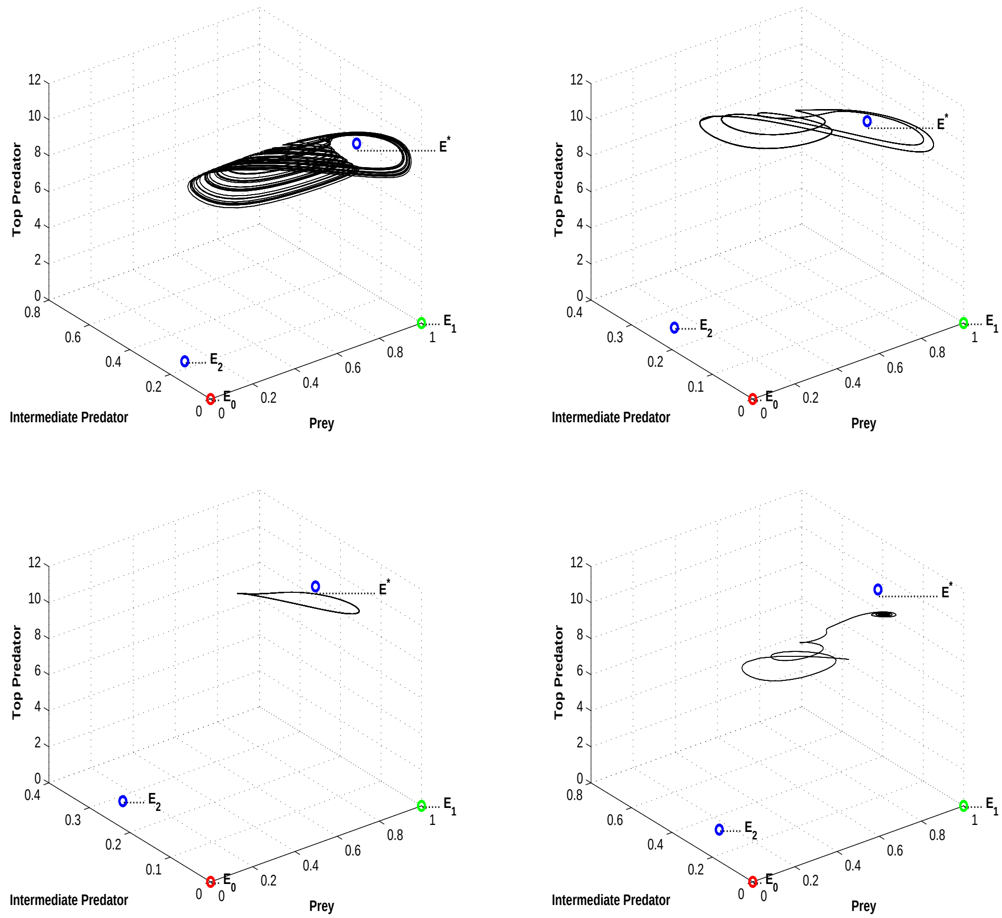

Figure 9: The Figures show the trajectory motion around the different equilibrium points (a) for $m_1 = 0.02$ and $m_2 = 0.03$ (b) for $m_1 = 0.08$ and $m_2 = 0.09$ (c) for $m_1 = 0.2$ and $m_2 = 0.16$ (d) for $m_1 = 0.32$ and $m_2 = 0.25$ and other values of the parameters are given in Figure 1.