TOPOLOGICAL CHARACTERIZATIONS OF QCADLs

1 Dr. A. LAKSHMANA 2 G. PRAKASAM BABU

1 Associate Professor in Mathematics and Principal
SRI CHANKYA DEGREE COLLEGE, VISAKHAPATNAM, A. P.
2 Sr. Lecturer in Mathematics
S. V. R. K. GOVT. DEGREE COLLEGE FOR MEN, NIDADAVOLE-534301, A.P
arnepallilakshmana@ gmail.com, prakash.g368@gmail.com

Abstract

We characterize a quasi-complemented in terms of topological terms and prove that if P is a prime ideal of I(L) (Q is a prime ideal of L), then C(P)= $\cup\{J \in I(L)/J \in P\}$ is a prime ideal in L ($\tau(Q) = \{J \in I(L)/J \subset Q\}$ is prime ideal in I(L). We prove that the necessary and sufficient conditions for an ADL in which every dense element is a maximal to become a quasi-complemented ADL in terms of τ_h^M, τ_d^M and prove that L is a quasi-complemented ADL if and only if M is a compact in the hull-kernel topology. and derive a necessary and sufficient condition for an ADLs to become a quasi-complemented ADLs.

AMS Subject classification : 06D99, 06D15.
Key words : Almost Distributive Lattices(ADLs), α-ideals, Annihilator Ideals in ADLs and quasi complemented ADLs.

1 Preliminaries

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy U M and Rao G C [4], as a common abstraction of existing lattice theoretic and ring theoretic generalization of Boolean algebra. The concept of quasi-complemented Almost Distributive Lattices was introduced in [2].

Definition 1.1. An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL) if it satisfies the following axioms:

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b =$ b
(4) $(a \vee b) \wedge a =$ a
(5) $a \vee (a \wedge b) =$ a
(6) $0 \wedge a = 0$ for all $a, b, c \in L$

In the following a partial order is defined on an ADL $(L, \vee, \wedge, 0)$.

Definition 1.2. Let L be an ADL and for any $a, b \in L$. Then we say that a is less than or equal to b and write $a \leq b$ if $a \wedge b = a$ or equivalently $a \vee b = b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except possibly the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L, a distributive lattice.

Definition 1.3. Let $(L, \vee, \wedge, 0)$ be an ADL. By an interval in L we mean the set $[a, b] := \{x \in L \ / \ a \leq x \leq b\}$ for some $a, b \in L$ with $a \leq b$. Every interval $[a, b]$ in an ADL is a bounded distributive lattice. An ADL $(L, \vee, \wedge, 0)$ is said to be relatively complemented if every interval $[a, b]$, $a \leq b$ in L is a Boolean algebra.

Theorem 1.4. *Let L be an ADL and $a, b, c \in L$. Then we have the following*
1. $a \vee b = a \Leftrightarrow a \wedge b = b$
2. $a \vee b = b \Leftrightarrow a \wedge b = a$
3. $a \wedge b = a \wedge b$ *, whenever* $a \leq b$
4. $\wedge$ *is associate in* L
5. $a \wedge b \wedge c = b \wedge a \wedge c$
6. $(a \vee b) \wedge c = (b \vee a) \wedge c$
7. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
8. $a \wedge a = a$ *and* $a \vee a = a$

A non-empty subset I of an ADL L is called an ideal (filter) of L if $a \vee b \in I$ $(a \wedge b \in I)$ and $a \wedge x \in I$ $(x \vee a \in I)$, for any $a, b \in I$ and $x \in L$. If I is an ideal of L and $a, b \in L$, then $a \wedge b \in I \Leftrightarrow b \wedge a \in I$. The set I(L) of all ideals of L is a complete distributive lattice with the least element (0] and the greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I, J and the supremum is given by $I \vee J = \{i \vee j / i \in I, \ j \in J\}$. For any $a \in L$, (a] $= \{a \wedge x / x \in L\}$ is the principal ideal generated by a. Similarly, [a) $= \{x \vee a / x \in L\}$ is the principal filter generated by a. The set PI(L) of all principal ideals of L is sub lattice of I(L).

An element $a \in L$ is called dense if $(a]^* = (0]$ and set of all dense elements is denoted by D. Then D is filter, whenever D is non-empty. An ADL L with 0 is called a $*$-ADL if to each $x \in L$, there exists $y \in L$ such that $[x]^{**} = [y]^*$. An ADL L with 0 is a $*$-ADL if and only if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. Every $*$-ADL possesses a dense element.

Lemma 1.5. *Let L be an ADL with 0 and $a, b \in L$.Then we have the following :*
1. $(a] \vee (b] = (a \vee b] = (b \vee a]$
2. $(a] \cap (b] = (a \wedge b] = (b \wedge a]$

2 Topological Characterization of QCADLs

In this section, we characterize a quasi-complemented in terms of topological terms and prove that if P is a prime ideal of I(L) (Q is a prime ideal of L), then C(P)= $\cup\{J \in I(L)/J \in P\}$ is a prime ideal in L ($\tau(Q) = \{J \in I(L)/J \subset Q\}$ is prime ideal in I(L). We prove that the necessary and sufficient conditions for an ADL in which every dense element is a maximal to become a quasi-complemented ADL in terms of τ_h^M, τ_d^M and prove that L is a quasi-complemented ADL if and only if M is a compact in the hull-kernel topology. We prove that in an ADL with a maximal element m in which every dense element is maximal, then L is a quasi-complemented ADL if and only if $C(P) \in M$, for each $P \in \mathbf{m}(I(L))$ and moreover $J^{**} \in A_0(L)$, for each $J \in I(L)$ if and only if L is a quasi-complemented ADL and for each $Q \in M$, there exists unique $P \in \mathbf{m}(I(L))$ such that C(P) = Q. We derive the set of an equivalent conditions for an ADL in which every dense element is a maximal to become a quasi-complemented ADL in terms of annulets.

From the definitions of C(P) and $\tau(Q)$, we prove the following.

Lemma 2.1. *Let L be an ADL with 0. If P is a prime ideal in I(L), C(P) is a prime ideal in L.*

Proof. Let $x,\ y \in C(P)$. Then by definition of C(P), $x \in I_1,\ y \in I_2,$ for some $I_1,\ I_2 \in P$. It follows that $x \vee y \in I_1 \vee I_2$ and hence $I_1 \vee I_2 \in P$. Therefore $x \vee y \in C(P)$. Let $x \in C(P)$ and $y \in L$. Then $x \in I$ for some $I \in P$. It follows that $x \wedge y \in I$ and hence $x \wedge y \in C(P)$. Thus C(P) is an ideal in L. Let $x,\ y \in L$ such that $x \notin C(P)$ and $y \notin C(P)$. Suppose $x \wedge y \in C(P)$. Then $x \wedge y \in I$ for some $I \in P$. It follows that $(x \wedge y] \subseteq I$ and $I \in P$

$$\Rightarrow (x \wedge y] \in P$$
$$\Rightarrow (x] \wedge (y] \in P \quad (\text{ since } (x \wedge y] = (x] \wedge (y])$$
$$\Rightarrow (x] \in P \text{ or } (y] \in P \quad (\text{since P is a prime ideal})$$
$$\Rightarrow x \in C(P) \text{ or } y \in C(P) \quad (\text{since by definition of C(P)})$$

Which is a contradiction to $x \notin C(P)$ and $y \notin C(P)$. Therefore $x \wedge y \notin C(P)$. Hence C(P) is a prime ideal in L. ∎

Lemma 2.2. *Let L be an ADL with 0. If Q is a prime ideal in L, then $\tau(Q)$ is a prime ideal in I(L).*

Proof. Suppose Q is a prime ideal in L. Let $J_1,\ J_2 \in \tau(Q)$. Then by definition of $\tau(Q)$, $J_1 \subseteq Q$ $J_2 \subseteq Q$. It follows that $J_1 \vee J_2 \subseteq Q$. Hence $J_1 \vee J_2 \subseteq \tau(Q)$. Let $J \in \tau(Q)$ and $K \in I(L)$. Then $J \subseteq Q$ and hence $J \cap K \subseteq Q$. Therefore $J \cap K \in \tau(Q)$. Hence $\tau(Q)$ is an ideal. Let $J_1,\ J_2 \in I(L)$ such that $J_1 \cap J_2 \in \tau(Q)$. Then $J_1 \cap J_2 \subseteq Q$. It follows that $J_1 \subseteq Q$ or $J_2 \subseteq Q$ (since Q is a prime ideal in L). Therefore $J_1 \subseteq \tau(Q)$ or $J_2 \subseteq \tau(Q)$. Hence $\tau(Q)$ is a prime ideal in I(L).

∎

Now, we prove the following.

Lemma 2.3. *Let L be an ADL such that $C(P) \in M$, for each $P \in \boldsymbol{m}(I(L))$. Then the mapping $\phi : m(I(L)) \to M$ defined by $\phi(P) = C(P)$, for each $P \in \boldsymbol{m}(I(L))$ is an onto continues closed mapping.*

Proof. Clearly ϕ is well defined mapping. Let $Q \in \mathbf{m}(I(L))$. Then $\tau(Q)$ is a prime ideal in I(L) and it contains a minimal prime ideal P of I(L). Now, we shall prove that C(P) = Q. Let $x \in C(P)$. Then $x \in J$ for some $J \in P$. It follows that $(x] \subseteq J$ and $J \in P$. Therefore $(x] \in P$, since P being an ideal in I(L). Thus $(x] \in \tau(Q)$, since $P \subseteq \tau(Q)$. Hence $C(P) \subseteq Q$. We have C(P) and Q are both minimal prime ideals in L, we get C(P) = Q. Therefore ϕ is an onto mapping. Claim: $\phi^{-1}(M_a) = M_I$.
Now, Let $a \in L$. Then

$$\begin{aligned}\phi^{-1}(M_a) &= \{P \in \mathbf{m}(I(L)) \ / \ \phi(P) \in M_a\}\\ &= \{P \in \mathbf{m}(I(L)) \ / \ C(P) \in M_a\}\\ &= \{P \in \mathbf{m}(I(L)) \ / \ a \notin C(P)\}\\ &= \{P \in \mathbf{m}(I(L)) \ / \ (a] \subsetneq C(P)\}\\ &= \{P \in \mathbf{m}(I(L)) \ /(a] \notin P\}\\ &= M_{(a]}.\end{aligned}$$

Thus the inverse image of a basic open set in M is again a open set in $\mathbf{m}(I(L))$. Hence ϕ is a continuous map. The space $\mathbf{m}(I(L))$ is a compact space and M is a Hausedorff space. Hence the mapping ϕ being continuous, is a closed mapping. ■

We now derive the necessary and sufficient conditions for an ADL in every dense element is a maximal to become a quasi-complemented ADL.

Theorem 2.4. *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if for each $x \in L$, there exists $y \in L$ such that $M_x = h_M(y)$.*

Proof. Suppose L is quasi-complemented ADL. We have every quasi-complemented ADL is a $*$ ADL. Let $x \in L$. Then there exists $y \in L$ such that $[x]^* = [y]^{**}$. Therefore $h_M([x]^*) = h_M([y]^{**})$ and hence $h_M([x]^*) = h_M(y)$. Thus $M_x = h_M(y)$. Conversely suppose that for each $x \in L$, there exists $y \in L$ such that $M_x = h_M(y)$. We shall prove that L is quasi-complemented ADL. Since $M_x = h_M(y)$, $h_M([x]^*) = h_M([y]^{**})$. Hence we get $[x]^* = [y]^{**}$. Therefore $x \wedge y = 0$ and $x \vee y$ is dense. It follows that $x \wedge y = 0$ and $x \vee y$ is a maximal. Thus L is a quasi-complemented ADL. ■

Theorem 2.5. *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is a quasi-complemented ADL if and only if M is a compact in the hull-kernel topology.*

Proof. Suppose L is quasi-complemented ADL. Then for each $x \in L$ there exists $y \in L$ such that $M_x = h_M(y)$ and hence M_x is a basic closed set in M. Let $\{M_x / x \in \Delta\}$ be a family of closed sets in M with finite intersection property for some $\Delta \subseteq L$. Let F be a filter in L generated by Δ. Then for any $x_1,\ x_2,\ ....,\ x_n \in \Delta,\ \bigcap_{i=1}^{n} M_x \neq \phi$ and hence $M_{\bigwedge_{i=1}^{n} x_i} \neq \phi$. It follows that $\bigwedge_{i=1}^{n} x_i \neq 0$. Therefore $0 \notin F$ and hence F is a proper filter of L. It follows that F is contained in a maximal filter say K of L. Therefore $L-K$ is minimal prime ideal of L. Let $x \in \Delta$. Then $x \notin L-K$. Therefore $L-K \in M_x$, for all $x \in \Delta$. Hence $L-K \in \bigcap_{x\in\Delta} M_x$, we get $\bigcap_{x\in\Delta} M_x \neq \phi$. Thus M is compact in hull-kernel topology.

Conversely suppose M is a compact in the hull-kernel topology on M and $x \in L$. Then $h_M(x)$ being a closed subset of M, is a compact. If $x \in P$, then. Hence by Lemma 0.3.31, $[x]^* \nsubseteq P$. Thus $h_M(x) \cap h_M([x]^*) = \phi$. So that $h_M(x) \cap \bigcap_{t\in[x]^*} h_M(t) = \phi$. Now, $\{h_M(x) \cap h_M(t)/t \in [x]^*\}$ is a class of closed sets in $h_M(x)$ having empty intersection, there exists $t_1\ , t_2\ , ...,\ t_n \in [x]^*$ such that $h_M(x) \cap h_M(t_1) \cap h_M(t_2) \cap .... \cap h_M(t_n) = \phi$. Write $x' = \bigvee_{i=n}^{n} t_i$, then $h_M(x) \cap h_M(x') = \phi$. It follows that $M_x \cup M_{x'} =$ M and $M_x \cap M_{x'} = M_{x\wedge x'} = M_0 = \phi$. Therefore $M_{x'} = h_M(x)$ and $M_x = h_M(x')$. Hence $h_M([x]^{**})= h_M(x) = M_{x'} = h_M([x']^*)$. Hence by Lemma 0.3.46, we get $[x]^{**} = [x']^*$. It follows that $x\wedge x' = 0$ and $x\vee x'$ is dense. Thus $x\wedge x'= 0$ and $x\vee x'$ is a maximal and hence L is quasi-complemented ADL. ■

We immediately have the following from Theorems 2.4 and 2.5.

Corollary 2.6. *Let L be an ADL with maximal element m in which every dense element is a maximal. Then the following are equivalent:*
(1) L is a quasi-complemented ADL
(2) $\tau_h^M = \tau_d^M$
(3) M is compact in hull-kernel topology.

Recall that for a prime ideal P of I(L), C(P)= $\cup\{J \in I(L)/J \in P\}$ is a prime ideal in L and derive the following theorem.

Theorem 2.7. *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is a quasi-complemented ADL if and only if* $C(P) \in M$, *for each* $P \in \mathbf{m}(I(L))$.

Proof. Suppose L is a quasi-complemented ADL and $P \in \mathbf{m}(I(L))$. Clearly by Lemma 2.3, C(P) is a prime ideal in L. Let $x \in C(P)$. Then $x \in J$, for some $J \in P$. Therefore $(x] \subseteq J$ and hence $(x] \in P$. Since L is a quasi-complemented ADL, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x'$ is a maximal. But, $(x] \vee (x'] = (x \vee x'] =$ L and $(x] \vee (x'] \notin P$, since P is minimal prime ideal in I(L), it follows that $(x'] \notin P$. If $x' \in C(P)$, then $x' \in J$ for some $J \in P$. Therefore $(x'] \subseteq J$ and hence $(x'] \in P$, which is a contradiction. Hence $x' \notin C(P)$. Thus for each $x \in C(P)$, there exists $x' \notin C(P)$ such that $x \wedge x' = 0$. Hence C(P) is a minimal prime ideal of L. Hence $C(P) \in M$.

Conversely assume the condition. We have the mapping $\phi : \mathbf{m}(I(L)) \to M$ defined by $\phi(P) = C(P)$, for each $P \in \mathbf{m}(I(L))$ is onto continues and closed. Therefore M is a compact. By Theorem 2.5, we get L is a quasi-complemented ADL. ∎

We recall that the set $A_0(L)$ of all annulets of an ADL L forms a distributive lattice under the binary operations $\bar{\wedge}$ and $\veebar$ defined by $[x]^* \bar{\wedge} [y]^* = [x \vee y]^*$ and $[x]^* \veebar [y]^* = [x \wedge y]^*$, for any $[x]^*, [y]^* \in A_0(L)$.
Next, we characterize a quasi-complemented ADL in terns of annulets.

Theorem 2.8. *Let L be an ADL with a maximal element m in which every dense element is a maximal. Then $J^{**} \in A_0(L)$, for each $J \in I(L)$ if and only if L is a quasi-complemented ADL and for each $Q \in M$, there exists unique $P \in \boldsymbol{m}(I(L))$ such that C(P) = Q.*

Proof. Suppose $J^{**} \in A_0(L)$, for each $J \in I(L)$. Let $x \in L$. Then we have $[x]^{**} \in A_0(L)$. It follows that, there exists $x' \in L$ such that $[x]^{**} = [x']^*$. Hence $x \wedge x' = 0$ and $x \vee x'$ is a dense. Therefore by hypothesis, $x \wedge x' = 0$ and $x \vee x'$ is maximal. Thus L is a quasi- complemented ADL. Now, let $Q \in M$ and P_1 , $P_2 \in \mathbf{m}(I(L))$ such that $C(P_1) = C(P_2) =$ Q. Let $J \in P_1$. Then $J^* \in I(L)$ and $J \cap J^* = (0]$. Since $P_1 \in \mathbf{m}(I(L))$, $J \in P_1$, $J^* \notin P_1$, (by Lemma 0.3.31). Again, Since $J^* \in I(L)$, $J^* \in A_0(L)$. Hence there exists $y \in L$ such that $J^* = (y]^*$. Again, since $P_1 \in \mathbf{m}(I(L))$, $J \in P_1$ and $J^* \cap J^{**} = \{0\} \in P_1$, $J^{**} \in P_1$. Therefore $(y]^{**} \in P_1$, and hence $(y] \in P_1$. It follows that $y \in C(P_1) = C(P_2)$ and hence $(y] \in P_2$. Therefore $[y]^* \notin P_2$, since P_2 is a minimal prime ideal. Hence $J^* \notin P_2$. Therefore $J \in P_2$ and hence $P_1 \subseteq P_2$. Hence $P_1 = P_2$, since P_1, P_2 are minimal prime ideals. Therefore for each $Q \in M$, there exists unique $P \in \mathbf{m}(I(L))$ such that C(P) = Q.

Conversely, suppose L is a quasi-complemented ADL and for each $Q \in M$, there exists unique $P \in \mathbf{m}(I(L))$ such that C(P) = Q. If $C(P_1) = C(P_2)$, then $P_1 = P_2$, for P_1, $P_2 \in \mathbf{m}(I(L))$. Then there exists a mapping $\phi : \mathbf{m}(I(L)) \to M$ defined by $\phi(P) = C(P)$ is a homeomorphism. Let $J \in I(L)$. Then $h_M(J^*)$ is both open and closed sets in M. Hence $M - h_M(J^*)$ is compact(being closed subset of compact space M is compact). Therefore $M - h_M(J^*) = \bigcup_{i=1}^{n} M_{a_i} = M_{\bigvee_{i=1}^{n} a_i}$. Now, put $y = \bigvee_{i=1}^{n} a_i$. Then $M - h_M(J^*) = M_y = h_M([y]^*) = M - h_M([y]^{**})$. Hence $h_M(J^*) = h_M([y]^{**})$. It follows that $J^{**} \in A_0(L)$.

■

We conclude this section with the following.

Corollary 2.9. *Let L be an ADL with maximal element m. Then the following are equivalent:*
(1) M is compact, Hausdorff and extremally disconnected space.
(2) The space M and $\boldsymbol{m}(I(L))$ are homeomorphic.
(3)$J^{**} \in A_0(L)$, for each $J \in I(L)$
Further any of the above conditions implies that L is a quasi-complemented ADL.

References

[1] Birkhoff , G. : Lattice Theory , *Amer. Math. Soc. Colloq. Publ. XXV , Providence* (1967) , U.S.A.

[2] Cornish, W. H. :*Quasi-complemented lattice* , Comment. Math. Uni. Carolinae, 15 (1974), 501-511.

[3] Rao G.C. , Nanaji Rao G. , Lakshmana A : Quasi-complemented Almost Distributive Lattices, *Southest Asian Bulletin of Mathematics* ,Vol.No.39 (SEAMS-2015)

[4] Stone, M. H. : *Topological representation and brouwvrian logics*, Casopis. Pst. Mat. Fyz., 67(1937), 1-25.

[5] Swamy U.M., Rao G.C. : Almost Distributive Lattices , *Aust.Math.soc , Series A* , 31 (1981) , 77-91.